3 Sobolev Spaces

Exercise 3.1. Let $\Omega \subset \mathbb{R}^d$ be an open set. Show that the Sobolev space $W^{m,p}(\Omega)$ endowed with the norm

$$\|u\|_{W^{m,p}(\Omega)} = \sum_{|\alpha|\leq m} \|D^\alpha u\|_{L^p(\Omega)}$$

is a Banach space.

Exercise 3.2. Let $p \in [1,\infty)$ and $k \in \mathbb{N}$. Show that $W_0^{m,p}(\mathbb{R}^d) = W^{m,p}(\mathbb{R}^d)$.

Exercise 3.3. Let $\Omega = (-1,1) \times (0,1) \subset \mathbb{R}^2$. Define $u : \Omega \to \mathbb{R}$ by

$$u(x,y) = \begin{cases} 1 + \sin(xy) & \text{if } x \geq 0 \\ e^x & \text{if } x < 0. \end{cases}$$

Does u belong to $W^{1,p}(\Omega)$ for any $1 \leq p \leq \infty$?

Exercise 3.4. Let p be in $[1,\infty]$, $\Omega \subset \mathbb{R}^d$ be an open set and p' be the conjugate of p, namely $1 = \dfrac{1}{p} + \dfrac{1}{p'}$. Given $f \in W^{1,p}(\Omega)$ and $g \in W^{1,p'}(\Omega)$, show that $fg \in W^{1,1}(\Omega)$ and

$$\partial_{x_i}(fg) = g\,\partial_{x_i} f + f\,\partial_{x_i} g \qquad \text{a.e. in } \Omega,$$

where $\partial_{x_i}(fg)$, $\partial_{x_i} f$, $\partial_{x_i} g$ are the weak derivatives of fg, f, g respectively. Further, deduce that when $f, g \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ then $fg \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Exercise 3.5. Let $\Omega \in \mathbb{R}^d$ be an open set and $u : \Omega \to \mathbb{R}$ be a Lipschitz function with best Lipschitz constant L. Verify that the following extension of u

$$\widetilde{u}(x) = \begin{cases} u(x) & x \in \Omega \\ \inf\{u(y) + L|y-x| : y \in \Omega\} & x \in \mathbb{R}^d \setminus \Omega \end{cases}$$

is Lipschitz function with constant L on the whole $\mathbb{R}^d$.